Insight, chapter 5, 2.4 – 2.6

2.4 Transformations

Reference frames are multiple. There are as many personal reference frames as there are persons. And public reference frames are constructed of maps with "public markers," and since various markers can be used to orient the map, potentially, there are many public reference frames as well. Special reference frames are constructed with a point of origin (space 0,0 and time 0), with a particular orientation, and thus a shift in either the origin or the orientation will result in a new reference frame.

When two different reference frames exist, one can then ask how these are related to each other. The answer will allow one to transform reference frame one into reference frame two.

1. Particular transformations
    a. Personal Transformations: "Two men face each other, one may observe that the region of Space to the right of one man is to the left of the other, and so one concludes that under such circumstances what for one is 'right' for the other is 'left.'
    b. Public Transformations:
        i. Space: "In like manner, maps of different countries may be correlated by turning to the map of the continent that includes both countries." Or to globe.
        ii. Time: "Clocks in different positions may be synchronized by appealing to the earth's spin."
    c. Special Transformations
        i. Eg.
            1. Two special reference frames K and K'
            2. A point in K (x, y, z) is the same point in K' (x', y', z')
            3. One can then figure out the relationship of x to x', y to y' and z to z'. This relationship will be a mathematical equation x' = f(x). "f(x)" refers to an equation or function with x in it, and solving this equation results in x'.
            4. Try some examples with just two dimensions. Lonergan generalizes this to include a third and fourth dimension (z and t). He also incorporates the speed of light, which is an interesting constant that allows one to transform movements in space and time.

2.5 Generalize Geometry

Now Lonergan is going to make some general statements of transformations within any geometrical framework. Notice, that we usually think in terms of three dimensional special reference frames, perhaps with the fourth dimension of time added. However, one could add any number of "variables" up to an n-dimensional manifold. So instead of just "x,y,z,t" one could have "a, b, c … x, y, z, t." Lonergan is simply going to make some general statements about how one transposes an object (point, line, multi-dimensional object) from any reference frame to another.

I. The Principle:
   a. First, Lonergan gives the general formulation or definition of any function.
      i. $F(x_1, x_2, \ldots) = 0$
      ii. The reason that he equates this with zero, is because it then allows us to have all of the real "variables" or factors on the one side, which is/are what really constitute the function. So, in the equation "a + b = c" "a", "b" and "c" constitute the equation. Thus, to pull all of these on to the same side and thus symbolically "see" there direct relations, one can write "a + b – c = 0". No matter what the equation, one can always make it equal to zero. And thus, one can now formulate the general definition of any function as possessing some set of variables related to each other, and the relations as a whole equal zero.
   b. Then Lonergan asks, how can one transform one function (which, if geometrical, is embedded in a reference frame) into another function (which, if geometrical, is embedded in a reference frame). He wants to state the account of this that would be relevant for any transformations.
      i. Just as we had transposed x to x', y to y' above, so Lonergan is going to generalize this, and just say that any x can be transposed to any other x in another other x frame. Now, if the x axis, y axis, z axis, etc., etc., etc. have the same orientation to each other, then x' will be defined in relation to x directly. However, if the orientation is different, then one will need to include some of the other axises that help to "reorient" the entire equation. This is similar to the example above that places orients the x' axis in K' with the y axis in K.
      ii. Here is Lonergan's general formulation of the transformation is
         1. $X_1 = x_1(x'_1, x'_2, \ldots)$
         2. $X_2 = x_2(x'_1, x'_2, \ldots)$
         3. As a note, earlier, our x was our x_1 and our y was the x_2. Z in a three dimension frame, would be x_3, and time would be x_4 (the fourth variable).
      iii. Then Lonergan states the general formulation of the resultant transformed equation. If one substitutes the transformation into the original function, then one ends up in the new frame of reference--the new function.
         1. $G(x'_1, x'_2, \ldots) = 0$
            a. Notice, like the first equation "$F(x_1, x_2, \ldots) = 0$", it equals zero. And now, what one has in the variables are the "transformed variables" that were calculated from the transformation equations.

iv. Notice then, what is the invariant in all of this? What is the intelligibility of the geometrical object? The invariant is not in the particular formulation or mathematical expression tied to a particular frame of reference, since that changes when one shifts the frame of reference. Thus, what needs to take place is the finding of a mathematical expression that does not change its meaning when a transformation of a reference frame takes place. If this can be done, then one has found the "intelligibility" of the geometrical object as such, distinct from the particular frames of reference in which it is placed. Then one has found the true "form" within the particular functions.

c. Now, when one transforms the variables from one frame to another, it is the transformation equations, the proportionate variables in each frame "mean the same thing." Hence in our first example from the two dimensional transformations above, the points (x,y) and (x',y') mean the same thing. What is that meaning? The transformation equations express it. Likewise, for the more complex four dimension functions (x, y, z, t) and (x', y', z', t') all mean the same thing when speaking of the same geometrical object. The common intelligibility or form of this object is expressed in the transformation equations.

II. Two applications of the principle

a. What is the principle?"The mathematical expression of the principles and laws of a geometry will be invariant under the permissible transformations of that geometry.

b. Application 1: In discovering invariant mathematical expressions, which do not change their meaning through transformations, one will discover the true invariant intelligibilities of geometrical objects in different types of geometries. Euclidean transformation → affine transformations (vectors) → projective transformations (geometric objects as projected into another frame, eg. Shadows from a tree) → topological transformations (properties surface patterns, deformations, etc.).

c. Application 2: Riemannian manifolds.

2.6 A logical note

Different reference frames will result in different statements about the same geometrical objects. The difference therefore is a difference not in the object but in the discourse. Thus, the relations of discourse as discourse regard not the "things specified" but in the specifications of the discourse about the things specified. Thus, transformation equations do not directly regard geometrical objects, but rather the different ways for specifying of discoursing about geometrical objects. Thus invariance regards not the geometrical object as such, but of the discourse about that object.